代数

Thm. 4.29 GKdim $\mathcal{B}(V)$ is finite $\Leftrightarrow$ $a=1$ and $q_{22}=-1$;
in this case GKdim $\mathcal{B}(V)=2$

Lem 4.30 If $a \notin \mathbb{N}$, then GKdim $\mathcal{B}(V)=\infty$

Pf. $N \in \mathbb{N}$,
Claim: if $a \notin I_N$ then
$$f_{i_1} \cdots f_{i_k}, \quad k \in \mathbb{N}_0, \quad 0 \le i_1 < i_2 < \cdots < i_k < 2N+1$$
are linearly independent in $\mathcal{B}(V)$

$f_n = \mathrm{ad}_c\, x_1 (z_n)$

Since $f_i \in \mathcal{B}^{i+2}(V)$, Lem 2.5 applies.

[Lem 2.5 $B=\bigoplus B^n$ f.g graded alg. $B^0=\mathbb{k}$
$(y_k)_{k \ge 0}$ a family of homogeneous elts s.t
$$\{ y_{i_1} \cdots y_{i_2} : i_j \in \mathbb{N},\ i_1 < \cdots < i_l \} \text{ linearly indep.}$$
$\exists$ $m, p \in \mathbb{N}$ s.t
$$\deg y_i \le m i + p \quad \forall i \in \mathbb{N}. \Longrightarrow \mathrm{GKdim}\, B = \infty$$

$$\deg f_i = i+2$$

$$\partial_3(f_{i_1} \cdots f_{i_k}) = \partial_3(f_{i_1} \cdots f_{i_{k-1}})(g_2 \cdot f_{i_k}) + f_{i_1} \cdots f_{i_{k-1}}\, \partial_3(f_{i_k})$$

$$\partial_3(f_{2n}) = 2 \prod_{i=1}^{n} (i-a)\, x_{21}^n x_1$$

$p \ge 0$ $\quad \partial_1(x_2^c x_{21}^b) = b\, x_2^c x_{21}^b x_1 \qquad \partial_1(x_2^c x_{21}^b x_1) = x_2^c x_{21}^b$

$i_k = 2m_k$

$$\partial_1^{2m_k+1}(\quad) \longrightarrow b_k f_{i_1} \cdots f_{i_{k-1}}$$

Lem 4.32 If $a \ne 1$, $q_{22} \notin G_2' \cup G_3'$, then GKdim $\mathcal{B}(V)=\infty$.

Pf. Lem 4.28 & $a \ne 1 \Longrightarrow f_0, f_2 \ne 0$

[$\partial_3(f_0) = 2x_1$]

(4.53) $\quad ad_c\, x_1 (f_n) = 0 \quad \forall n \geq 0$

Lem 4.31

$$g_1 g_2 \cdot f_0 = q_{12} f_0 \,, \qquad g_1 g_2 \cdot f_2 = q_{21}^2 q_{22} f_2$$
$$g_1^3 g_2 \cdot f_0 = q_{12}^2 q_{22} f_0 \,, \qquad g_1^3 g_2 \cdot f_0 = q_{22} f_2$$

$W = \Bbbk f_0 + \Bbbk f_2$ is a braided vector subspace of $\Bbbk^!$

of diagonal type braid matrix $(P_{ij})_{i,j \in I_2}$

$$c(f_0 \otimes f_0) = f_{0(-1)} \cdot f_0 \otimes f_{0(0)} = g_1 g_1 \cdot f_0 \otimes f_0 = q_{22} f_0 \otimes f_0$$
$$c(f_0 \otimes f_2) = q_{21}^2 q_{22} f_2 \otimes f_0$$
$$c(f_2 \otimes f_0) = q_{12}^2 q_{22} f_0 \otimes f_2 \qquad (P_{ij}) = \begin{bmatrix} q_{22} & q_{21}^2 q_{22} \\ q_{12}^2 q_{22} & q_{22} \end{bmatrix}$$
$$c(f_2 \otimes f_2) = q_{22} f_2 \otimes f_2$$

$P_{11} = P_{22} = q_{22} \qquad P_{12} P_{21} = q_{22}^{\,2}$

Thus $\mathrm{GKdim}\, W = \infty$.

Indeed. if $q_{22} \notin G_\infty$ then W does not admit all reflections

[We can reflect V at $i \in I$ if $\quad j \neq i, \ \exists\, n \in \mathbb{N}_0$

s.t. $\qquad (n+1)_{q_{ii}} (1 - q_{ii}^n q_{ij} q_{ji}) = 0$

define a GCM (c_{ij}) by $c_{ii} = 2$

$$c_{ij} := -\min \{ n \in \mathbb{N}_0 \mid (n+1)_{q_{ii}} (1 - q_{ii}^n q_{ij} q_{ji}) = 0 \} \quad j \neq i$$

The reflection at vertex i of q is the matrix

$$R^i(q) = (t_{ij})_{j,k \in I}$$

$$t_{jk} := q_{jk}\, q_{ik}^{-c_{ij}}\, q_{ji}^{-c_{ik}}\, q_{ii}^{c_{ij} c_{ik}}$$

$R^i(V)$

We say V admits all reflection if we can reflect V at i_1, then we can reflect $R^{i_1}(V)$ at $i_2 \in I$ ....

[Rem 2.5 If $\mathrm{GKdim}\, B(V) < \infty$, then we can reflect V at every $i \in I$. Hence V admits all reflections.]

$$(n+1)_{p_{22}} (1 - p_{22}^n p_{21} p_{12}) = (n+1)_{q_{22}} (1 - q_{22}^n q_{22}^2) \neq 0$$

if $q_{22} \notin G_\infty$

$\Rightarrow$ W does not admit all reflections.

if $\mathrm{ord}\, q_{22} = N > 3$, then W is of Cartan type with Cartan matrix $\begin{bmatrix} 2 & 2-N \\ 2-N & 2 \end{bmatrix}$

by Thm 1.6 $\mathrm{GKdim}\, B(W) = \infty$

if $q_{22} = 1$ then $U := \Bbbk x_1 + \Bbbk x_3$ is a braided vector space of diagonal type with braid matrix $(p_{ij})_{i,j \in I_2}$

$$(p_{ij}) = \begin{bmatrix} \epsilon & q_{12} \\ q_{21} & q_{22} \end{bmatrix} = \begin{bmatrix} -1 & q_{12} \\ q_{21} & 1 \end{bmatrix} \qquad (4.1) \quad c(x_i \otimes x_j)$$

By Lem 2.8 $\mathrm{GKdim}\, B(W) = \infty$.

[Lem 2.8 if $\langle x_1, x_2 \rangle$ is a braided vector space of W of diagonal type, $q_{11} = 1$ $q_{12} q_{21} \neq 1 \Rightarrow \mathrm{GKdim}\, B(W) = \infty$]

#

Lem 4.33. If $q_{22} \in G_N'$, where $N \geq 3$, then $\mathrm{GKdim}\, B(V) = \infty$

pf. case 1, N odd.

B^{diag} is a braided graded Hopf alg. of diagonal type generated by $U := \sum_{i=3}^{3} \Bbbk x_i$. The braiding matrix (p_{ij}) of U satisfies

$$p_{11} = p_{13} p_{31} = p_{22} = p_{23} p_{32} = -1, \quad p_{12} p_{21} = 1, \quad p_{33} = q_{22}$$

[By (4.1), we get braiding matrix of U

$$(p_{ij}) = \begin{bmatrix} -1 & -1 & q_{12} \\ -1 & -1 & q_{12} \end{bmatrix} \qquad B^{dia} = \mathrm{gr}\, B$$

$$q_{21} \quad q_{21} \quad q_{22} \Big]$$

$V_1 \subset V_2 \subset V_3$

We ~~define~~ can define the reflection $R_3(U)$

C_{ij} :

$$C_{12} = -\min\{n \in \mathbb{N}_0 \mid (n+1)_{p_{11}} (1 - p_{11}^n p_{12} p_{21}) = 0\}$$

$$(n+1)_{t_{11}} (1 - (-1)^n) = 0 \Rightarrow n = 0$$

$$(C_{ij}) = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ 1-N & 1-N & 2 \end{bmatrix}$$

t_{jk} :

$$t_{11} = p_{11} p_{31}^{-c_{31}} p_{13}^{-c_{31}} p_{33}^{c_{31} c_{31}} = (-1)(-1)^{N-1} q_{22}^{1-N} = -q_{22}$$

$$(t_{ij})_{i,j \in I_2} = \begin{bmatrix} -q_{22} & -q_{22} \\ -q_{22} & -q_{22} \end{bmatrix}, \text{ affine type}$$

$$\begin{bmatrix} 2 & 2-2N \\ 2-2N & 2 \end{bmatrix}$$

$q_{22} \in G_N' \Rightarrow -q_{22} \in G_{2N}'$

$2N \geq 6$

$\operatorname{GKdim} \mathcal{B}(R_3(U)) = \infty$ by Thm 1.6.

$$\underline{\operatorname{GKdim} \mathcal{B}(V) \geq \operatorname{GKdim}} \mathcal{B}^{diag} \geq \operatorname{GKdim} \mathcal{B}(U) = \operatorname{GKdim} \mathcal{B}(R_3(U))$$

$=$ $\quad \operatorname{gr}\mathcal{B}$ $\quad \mathcal{B}$ $\quad$ Thm 2.1 $\quad = \infty$

$\operatorname{GKdim} \mathcal{B}(V) \geq \operatorname{GKdim} \mathcal{B}(\operatorname{gr} V)$

$\operatorname{GKdim} \mathcal{B}^{diag}$

Case 2. $N = 2M$

Then U as in the ~~the~~ proof above has generalized Dynkin diagram

$$\overset{-1}{\underset{1}{\circ}} \overset{-1}{\text{———}} \overset{q}{\underset{3}{\circ}} \overset{-1}{\text{———}} \overset{-1}{\underset{2}{\circ}}$$

This is of Cartan type $\quad A = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ -M & -M & 2 \end{bmatrix}$

if $n = 2$, $|A| = 0$ A is of affine type

$\Longrightarrow$ GKdim $B(V) = \infty$ by Thm 1.6.

if $n > 3$, then A contains a rank 2

submatrix of affine or indef. type.

$A_1 = \begin{bmatrix} 2 & -1 \\ -n & 2 \end{bmatrix}$. $|A_1| = 4 - n \leq 0$

if $n = 3$, then A is of hyperbolic type

$A = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ -3 & -3 & 2 \end{bmatrix}$. $\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$ finite type

$\Longrightarrow$ GKdim $B(V) = \infty$ by Hyperthesis 1.7 #

Lem 4.38 The set $B = \{ f_1^{n_1} f_0^{n_2} z_1^{n_3} z_0^{n_4} : 0 < n_i \leq 1,\ i \in I_4 \}$

is a basis of K. $\dim K = 16$

$(a = 1,\ q_{22} = -1)$

GKdim $B(V) = 2$

$\|$

GKdim K + GKdim $B(V_1)$

$\|$ $\|$

0 2

pf. K is spanned by $f_1^{n_1} f_0^{n_2} z_1^{n_3} z_0^{n_4}$

$n_i \in \mathbb{N}_0$.

By (4.49) $K' = \langle z_m, f_m : m \in \mathbb{N}_0 \rangle$

K is generated as an alg by z_i's f_i's.

By (4.63) $2 z_2 = f_1$

Lem 4.28 $\Longrightarrow$ $\partial_3(f_2) = 0$ $(a = 1)$ $\Longrightarrow$ $f_2 = 0$

it is enough to consider z_0, z_1, f_0, f_1

(4.67) $z_1 z_0 =$ $f_0 z_0 =$

(4.68) $f_1 z_0 =$ $f_0 z_1 =$

(4.69) $f_1 z_1 =$ $f_1 f_0 =$

B is linearly indep. :

$\partial_i(f_j) = \partial_i(z_j) = 0 \qquad i = 1, 2, \quad j = 0, 1$

$\partial_3(z_0) = 1 \qquad \partial_3(z_1) = 2x_2 + x_1 \qquad (4.54)$

$\partial_3(f_1) = 2x_1x_1 - 2x_2x_1$

Assume $\sum a_{n_1n_2n_3n_4} f_1^{n_1} f_0^{n_2} z_1^{n_3} z_0^{n_4} = 0$

$\partial_1\partial_2\partial_3(f_1f_0z_1z_0) = -4q_{12}^2 f_0 z_1 z_0$

$\partial_3\partial_1\partial_3(f_0z_1z_0) = -4x_2^2 \neq 0$

$\Rightarrow \underline{f_1f_0z_1z_0} \neq 0 \qquad \deg f_1f_0z_1z_0 = 8$

$\Rightarrow a_{1111} = 0$

$\deg f_1f_0z_1 = 7 \qquad \cancel{f_1} f_1 f_0 z_1 \neq 0$

$\Rightarrow a_{1110} = 0$

$a_{1101}f_1f_0z_1 + a_{1011}f_1z_1z_0 = 0$

$0 = \underline{f_0}(a_{1101}f_1f_0z_1 + a_{1011}f_1z_1z_0)$

$= -q_{21}a_{1011}\underbrace{f_1f_0z_1z_0}_{\neq 0}$

$\neq 0$

$(4.61) \quad f_0^2 = 0$

$\Rightarrow a_{1011} = 0$

$0 = z_1(0 \qquad) = -q_{21}(\underline{a_{1011}} + \underline{a_{1101}})\underline{f_1f_0z_1z_0}$

$\Rightarrow a_{1101} = 0$

Similarly, all $a_{n_1n_2n_3n_4} = 0 \Rightarrow B$ is a basis of K

\#

prop 4.40 The alg $B(C_1)$ is presented by generators x_1, x_2, x_3 and relations (3.7) (3.8) (4.58) (4.59) (4.60) (4.61) (4.62)

$\mathcal{B} = \{x_1^{m_1} x_{21}^{m_2} x_2^{m_3} f_1^{n_1} f_0^{n_2} z_1^{n_3} z_0^{n_4} : m_1, n_i \in \{0,1\},$
$m_2, m_3 \in \mathbb{N}_0\}$

is a basis of $\mathcal{B}(C_1)$ GKdim $\mathcal{B}(C_1) = 2$.

Pf. These relations hold in $\mathcal{B}(C_1)$

Then the quotient $\widetilde{\mathcal{B}}$ of $T(V)$ by these relations proj. onto $\mathcal{B}(C)$

claim $\underline{I}$ spanned by $\mathcal{B}$ is a right ideal of $\widetilde{\mathcal{B}}$.

$\underline{I}\, x_3 \subset \underline{I}$ $\quad (x_3 = z_0)$

$\underline{I}\, x_1 \subset \underline{I}$, $\quad \underline{I}\, x_2 \subset \underline{I}$ by Lem 4.37

Rem. 4.36

Since $1 \in \underline{I}$ $\widetilde{\mathcal{B}}$ is spanned by $\mathcal{B}$

To prove $\widetilde{\mathcal{B}} \simeq \mathcal{B}(C_1)$. it remains to show $\mathcal{B}$ is linearly independent. in $\mathcal{B}(C_1)$

$\underline{x_1^{m_1} x_{21}^{m_2} x_2^{m_3}} f_1^{n_1} \cancel{f_0}^{n_2} z_1^{n_3} z_0^{n_4}$ linearly indep

$\Longleftrightarrow$ $\underline{\underline{x_2^{m_3} x_{21}^{m_2} x_1^{m_1}} f_1^{n_1} f_0^{n_2} z_1^{n_3} z_0^{n_4}}$ . . . . . . $(*)$

$[\quad x_{21}^n x_2 = x_2 x_{21}^n - n x_1 x_{21}^n, \quad x_1 x_{21} = x_{21} x \,]$

P20 $\quad \partial_1 (x_2^c x_{21}^b x_1) = x_2^c x_{21}^b$

$\partial_1 (x_2^c x_{21}^b) = b\, x_2^c x_{21}^b x_1$

$\partial_1 (\quad * \quad) = \underline{\partial_1 (x_2^{m_3} x_{21}^{m_2} x_1^{m_1})} (g_1 \cdot f_1^{n_1} \cancel{f_0}^{n_2} z_1^{n_3} z_0^{n_4})$

$\underline{+\, *} \quad \underset{\underset{0}{\parallel}}{\underline{\partial_1 (f_1^n f_0^n z_1^n z_0)}}$

$\partial_1^{2m_2} (\, * \,) = m_2!\, x_2^{m_3} x_1^{m_1}$

$\partial_1^{m_3 - 1} \partial_2 (x_2^{m_3}) \neq 0 \quad$ (See P20)

$\bigcap \ker \partial_i = \mathbb{K}$

$\cdots \partial_i (\quad) = k$

$\mathcal{B}(V)$

Lem 4.34 If $q_{22} = -1$, $a \neq 1$ then $\mathrm{GKdim}\, \mathcal{B}(V) = \infty$.

pf. $\partial_3(z_0) = 1$ $\quad \partial_3(z_1) = 2x_2 + a x_1$

$$\partial_3(z_2) = (2-a)x_1x_2 - a x_2 x_1$$
$$= 2x_1x_2 - a x_{21}$$

$\Rightarrow z_2 \neq 0$ in $\mathcal{B}(V)$

Assume $a \neq 1$ and $z_2 = ad_c x_2^2(x_3) = 0$ in $\mathcal{B}^{diag}$

Since $\mathcal{B}(V) \simeq K \# \mathcal{B}(V_1)$

$z_2 \in K \overset{?}{=} \ker \partial_1 \cap \ker \partial_2$ $\qquad K = \mathcal{B}(K')$

$\Rightarrow \exists \lambda \in \mathbb{k}^\times$ s.t. $\qquad K' = \langle z_m, f_m, m \in \mathbb{N}_0 \rangle$

$z_2 = \lambda f_1$ in K $\qquad \deg z_1 = \deg f_1 = 3$

$\partial_3(f_1) = u(2x_1 x_2 - a x_{21})$

Lem 4.28 $\quad \partial_3(f_1) \neq 2(2x_1x_2 - x_{21})$

$z_2 = ad_c x_2^2(x_3)$ is nonzero primitive elt in $\mathcal{B}^{diag}$ of

Γ-degree $g_1^2 g_2$

$\mathrm{GKdim}\,\mathcal{B}(V) = \mathrm{GKdim}\, \mathcal{B}^{diag} = \mathrm{GKdim}\, \widetilde{\mathcal{B}}_1 \geq \mathrm{GKdim}\, \widetilde{\mathcal{B}_2} = \infty$

Dynkin diagram.

$\widetilde{\mathcal{B}}_2 =$ the Nichols alg quotient of $\widetilde{\mathcal{B}}_1$

$\langle x_1 \; x_2 \; x_3 \; z_2 \rangle = V \qquad \mathcal{B}(V) = \widetilde{\mathcal{B}_2}$